# Dislocations and torsion in graphene and related systems

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A continuum model to study the influence of dislocations on the electronic properties of condensed matter systems is described and analyzed. The model is based on a geometrical formalism that associates a density of dislocations with the torsion tensor and uses the technique of quantum field theory in curved space. When applied to two-dimensional systems with Dirac points like graphene we find that dislocations couple in the form of vector gauge fields similar to these arising from curvature or elastic strain. We also describe the ways to couple dislocations to normal metals with a Fermi surface.

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## I. INTRODUCTION

Since the experimental synthesis of graphene the interest in this material has mostly focussed on its electronic properties [1]. As time goes by and as the experiments become more accurate the interest is being displaced to the morphological aspects and their possible influence on the electronic properties [2]. Part of the interest on these issues is prompted by the necessity to find the mechanisms limiting the mobility of the graphene samples [3], an important issue for the potential technological applications. Another source of interest lies on the special mechanical and elastic properties of graphene recently tested in experiments [4, 5, 6, 7, 8] and on the fascinating possibility to explore the physics of real two dimensional crystals.

Local disorder in graphene has been thoroughly studied and we refer to the review article [1] for a fairly complete list of references. A different type of disorder is provided by the observation of ripples in suspended graphene [9, 10] and in graphene grown on a substrate [11, 12]. The elastic and mechanical properties of graphitic structures have been studied intensely in the past, mostly in the context of understanding the formation of fullerenes and nanotubes. Very little work has been done for the flat graphene sheet [13, 14] and topological defects have been often excluded in these studies. In the fullerene literature it was established that the formation of topological defects (substitution of a hexagonal ring by other polygons) is the natural way in which the graphitic net heals vacancies and other damages produced for instance by irradiation [15]. Among those, disclinations (isolated pentagon or heptagon rings), dislocations (pentagon-heptagon pairs) and Stone-Wales (SW) defects (special dislocation dipoles) were found to have the least formation energy and activation barriers. Dislocations and SW defects have been observed in carbon structures [16] and are known to have a strong influence on the electronic properties of nanotubes. Dislocations in irradiated graphitic structures have been described in [17] and experimental observations have been reported in graphene grown on Ir in [18].

In previous works we used a general relativity formalism to study the influence of curved portions of the lattice on the electronic properties of graphene by coupling the Dirac fermions to the corresponding curved space. We found that curvature generates a fictitious gauge field - a result that is also obtained in more conventional descriptions - and, in addition, we predicted a space-dependent parameter in the kinetic term (similar to a space-dependent Fermi velocity) that depends on the intrinsic curvature of the sample and has non-trivial effects on the density of states [19, 20, 21, 22] and on the conductivity [23]. In the present work we extend this formalism to include a finite density of dislocations. Dislocations are modelled by adding torsion to the graphene sheet either curved or flat.

Torsion was first considered by Cartan and then by Sciama and Kibble in order to deal with spin in General Relativity [24]. Within this scheme the spin of a particle turns out to be related to the torsion just as its mass is related to the curvature. Torsion is generally ignored in the context of GR, because the torsion field is algebraic in the Einstein-Cartan theory (its equations of motion are constraints, it has no dynamics) and as a consequence there is no torsion in the absence of matter. When matter generating torsion is added, its observable effects are suppressed by the smallness of the gravitational coupling and they are negligible [25]. These two drawbacks are overcome when torsion is included in a geometric fashion as associated to dislocations. In the context of electron systems like graphene the dynamics of the lattice defects occurs at a much higher energy than the electronic processes (in graphene, lattice processes are related to the sigma bonds and have typical energies of the order of tens of eV while the continuum relevant low energy processes are of few tens of meV). It then makes perfect sense to consider the motion of electrons in a frozen geometry and hence the fact that torsion has no dynamics is not a problem. Moreover the suppression of the coupling of the torsion and electronics degrees of freedom will be of the order of magnitude of the quotient between electronic

and elastic degrees of freedom that, although small, will be much bigger than the gravitational scale.

The purpose of these notes is twofold: First we describe a continuum description that allows to explore the effects of dislocations on the electronic properties of graphene and other materials, and, second, we propose an "analog" model that amplifies the effect of torsion in a General Relativity context.

## II. DISLOCATIONS AND TORSION

Topological defects in crystal lattices can be classified in two kinds: Dislocations and disclinations. A dislocation can be thought of being formed by performing a cut in a bulk material and introducing extra rows of atoms (Volterra process). These types of point defects can be easily identified by going around a closed circuit enclosing the end of the line cut with discrete lattice steps. Without defects the path will end up at the same point. In the presence of a dislocation it ends up somewhere else and an extra vector has to be added to close the path. This is called the Burgers vector. Dislocations can be further classified by the direction of the Burgers vector with respect to the plane defined by the circuit: In an edge dislocation the Burgers vector lies in this plane, while in screw dislocation it is perpendicular to it. Fig. 1 shows an edge a and screw dislocation in the square lattice characterized by a Burgers vector parallel (perpendicular) to the displacement. A disclination is similarly formed by adding a wedge of atoms, and it can be identified by computing the total angle subtended by a closed path enclosing the defect (Frank angle). A disclination dipole can also be seen as a dislocation. ¿From the point of view of geometry, dislocations represent a source of translational mismatch "anholonomy",(see Appendix A 2 for a rigorous description of torsion), and as we will see, are very naturally described in the continuum by a connection with torsion. In differential geometry, a connection defines the notion of parallel transport. When a vector $B^\mu$ is parallel transported from $x^\mu$ to $x^\mu + dx^\mu$, it experiences an infinitesimal variation given by

$$\delta B^\mu = -\Gamma^\mu_{\nu\rho}(x)B^\nu dx^\rho. \qquad (1)$$

The equivalent of the closed circuit in the definition of the Burgers vector can be defined by taking two infinitesimal vectors $m^\mu$ and $n^\nu$, and trying to build a parallelogram with them. For this purpose, one parallel transports the vector $m^\mu$ along $n^\nu$:

$$m'^\rho = m^\rho + n^\rho - \Gamma^\rho_{\mu\nu}m^\mu n^\nu, \qquad (2)$$

and the vector $n^\nu$ along $m^\mu$

$$n'^\rho = m^\rho + n^\rho - \Gamma^\rho_{\nu\mu}m^\mu n^\nu. \qquad (3)$$

As depicted in Fig. II, if there is a dislocation in the region enclosed by the path, the parallelogram obtained
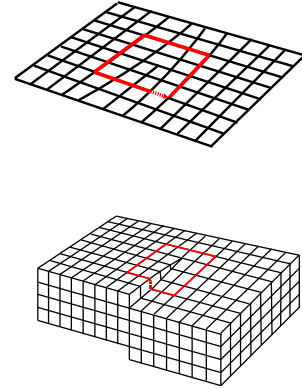


FIG. 1: (Color online) Upper: Edge dislocation in a square two-dimensional lattice. The Burgers vector is parallel to the displacement. Down: Screw dislocation in a three dimensional cubic lattice. The Burgers vector is perpendicular to the displacement.

does not close, and the part that is missing is proportional to the antisymmetric part of the connection, defined as the torsion:

$$m'^\rho - n'^\rho = \left[\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right] m^\mu n^\nu \equiv T^\rho_{\mu\nu}m^\mu n^\nu \qquad (4)$$

Notice that the two paths would have closed in a curved space with a standard affine connection.

In the continuum limit, one can describe a density of dislocations by the density of Burgers vector, and we see that torsion plays precisely this role. Hence the natural identification of torsion with dislocations. The connec-
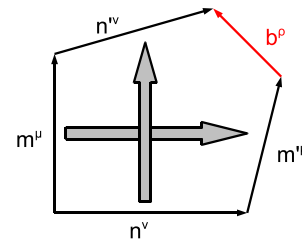


FIG. 2: Non-closure of infinitesimal parallelograms due to the presence of torsion

tion of torsion with the continuum theory of crystal dislocations goes back to Kondo [26] and has been formalized in [27, 28]. Various aspects of the problem have been explored in [29]. A nice review on the relation of gravity with topological defects in solids is [30]. The geometric approach to defects in solids [28, 31] relates the metric
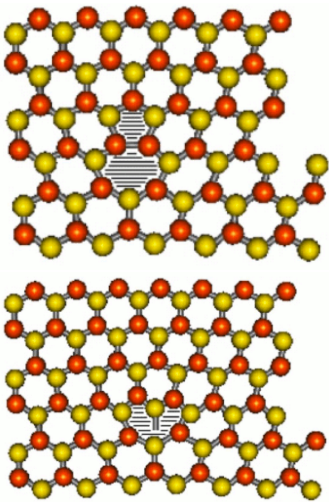
FIG. 3: (Color online) Structure of the glide (up) and shuffle (down) dislocations in the planar graphene lattice.

of the curved -crystalline- surface with the deformation tensor and establishes that disclinations are associated to the curvature tensor and a finite density of dislocations generates a torsion term.

In flat surfaces screw dislocations do not exist. Dislocations in graphene are made of pentagon–heptagon pairs and they have been widely studied in connection with the properties of carbon nanotubes [32] and, more recently, in the flat graphene sheets or ribbons [33, 34, 35]. Observations of topological defects in graphene have been reported in [16, 36, 37]. The stability of dislocations in graphene has been explored recently in [34] and two types of stable dislocations were found. The most common are the so-called glide dislocations made by a pentagon-heptagon pair shown in the upper part of Fig. II. In addition we can have shuffle dislocations (lower part of Fig. II made by a octagon with a dangling bond. Both are edge dislocations and add one or several lines of atoms to the lattice. The difference is that shuffle dislocations contain an extra atom with respect to the glide dislocations. In an infrared continuum formulation both are seen as point defects that do not disturb the lattice out of the core. The difference with conical defects as pentagons or heptagons is that when an electron circles such a defect it will found itself displaced from the original position a distance proportional to the Burgers vector as shown in Fig. II.

## III. COUPLING DIRAC FERMIONS TO A CURVED SPACE WITH DISLOCATIONS

The model that we propose to study the low energy properties of graphite and graphene in the presence of lattice defects (disclinations and dislocations) is based on the assumption that the low–energy description of

the quasiparticles in terms of massless Dirac fermions remains valid in the presence of a diluted distribution of defects. The stability of the Fermi points in multilayer arrangements of the honeycomb lattice has been demonstrated in [38]. As we have described in the previous section the presence of dislocations in a crystal can be described in the continuum limit by adding torsion to the space spanned by the lattice. We will here follow a covariant approach close to the general relativity formulation what allows to consistently couple matter to geometry. This description began with the early paper of ref. [39] where the Dirac equation on a sphere was used to solve the electronic spectrum of fullerenes. The pentagonal defects (disclinations) were described by a fictitious gauge field. The formalism was later generalized to study the influence of disclinations [40, 41] and of smooth curvature [21, 22] on the density of states of graphene. These previous works showed that the covariant formulation allows to obtain sensible predictions that are in agreement with the physical observations when the infrared properties are considered. In particular it was shown that the curvature induces a fictitious vector gauge field. We will here analyze the nature of the effective gauge fields created by the presence of dislocations within the covariant framework.

Before introducing the changes induced by having torsion we summarize the situation in a curved space.

The dynamics of a massless Dirac spinor in a curved spacetime is governed by the modified Dirac equation (see Appendix A):

$$i\gamma^\mu(\mathbf{r})D_\mu\psi = 0 \qquad (5)$$

where $\gamma^\mu$ are the curved space $\gamma$ matrices that depend on the point of the space and can be computed from the generalized anticommutation relations

$$\{\gamma^\mu(\mathbf{r}), \gamma^\nu(\mathbf{r})\} = 2g^{\mu\nu}(\mathbf{r}),$$

and the covariant derivative operator is defined as

$$D_\mu = \partial_\mu - \Omega_\mu,$$

where $\Omega_\mu$ is the spin connection of the spinor field whose physical implications have been discussed in [21, 22].

Coupling a Dirac field to a curved space with torsion gives the so-called Einstein-Cartan-Dirac theory. The details of the derivation of the Dirac equation in a space with torsion are given in Appendix A 4. Here we will simplify the description putting it in a different, more physical way. The minimal coupling of any geometrical or physical fields to the Dirac spinors adopts always the form of a covariant derivative:

$$\partial_\mu \Rightarrow D_\mu = \partial_\mu + A_\mu, \qquad (6)$$

where the given vector can be an electromagnetic potential induced by a real electromagnetic field or any other real or fictitious gauge field associated to deformations or to geometrical factors [42]. In the case of having a

density of dislocations in the graphene sheet modelled by torsion we can construct two potential candidates to gauge fields. In four dimensions, from the rank three torsion tensor (A14) $T_{\mu\nu\rho}$ the following vectors can be built:

$$V_\mu = g^{\nu\rho}T_{\nu\rho\mu} \ , \ \ S_\mu = \epsilon_{\mu\nu\rho\sigma}T_{\nu\rho\sigma}. \qquad (7)$$

The field $V_\mu$ is a real vector and can be associated to the density of edge dislocations while $S_\mu$ is an axial (pseudo) vector associated to the density of screw dislocations. These fields couple to the vector and axial current density respectively so the full Lagrangian is:

$$L_{\text{int}} = \int d^4x \bar{\Psi} \left[\gamma^\mu(\partial_\mu + ieV_\mu + i\eta\gamma^5 S_\mu)\right]\Psi, \qquad (8)$$

where $e$ and $\eta$ are coupling constants related to the density of edge and screw dislocations respectively. The physical consequences of the vector torsion are then similar to these coming from curvature or elasticity. The coupling of the spinor to the axial field $A_\mu$ can have important consequences as it breaks time reversal symmetry.

### A. Two dimensional systems

In (2+1) dimensions the completely antisymmetric part of the torsion tensor is proportional to the Levi-Civita tensor:

$$T_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}\Phi, \qquad (9)$$

where $\Phi$ is a pseudoscalar field.

In 2+1 dimensions there are no screw dislocations, simply because there is no third spatial dimension, and the antisymmetric part of the torsion is rather related to time displacements. Since the time components of the torsion generated by a dislocation are zero, this term vanishes always. The remaining coupling to the trace part acts a new vector gauge field with special symmetry properties associated to the dislocations structure of the sample. This field will add up to these coming from curvature or to real electromagnetic fields.

A remaining interesting question is that of how this extra gauge field couples to the electronic excitations around different Fermi points (valleys). It is known that the fictitious gauge fields arising from elastic deformations as well as these coming from curvature, give rise to effective magnetic fields pointing in opposite directions in different valleys. The reason is that the sign of the coupling of the effective vector fields depends uniquely on the definition of the spin connection (A6)

$$\Omega_\mu = \frac{1}{8}\omega_\mu{}^{ab}\left[\gamma_a, \gamma_b\right], \qquad (10)$$

and is determined by the product of gamma matrices in it. As the effective Hamiltonians around each of the Fermi points differ in the sign of only one of the gamma matrices (they are similar to parity conjugated) the spin connection has opposite signs in the two valleys. This is irrespective of whether the connection has or not an antisymmetric part and hence applies also to the coupling induced by torsion.

### B. Three dimensional systems supporting Dirac fermions

We have seen in the previous sections that the effects of dislocations in two dimensional crystals like graphene are – at most – describable in terms of a fictitious gauge field similar to the one associated to curvature. The formalism outlined in this work acquires full power when applied to three dimensional crystals supporting screw dislocations and having Dirac fermions in the spectrum. Graphite is one of the most obvious candidates where screw dislocations are very common [43] and elementary excitations behaving as Dirac fermions were predicted near the K points of the band structure in the early calculations [44, 45]. These Dirac fermions have been experimentally confirmed in [46, 47].

The other perfect candidates are the the topological insulators [48, 49] specially the $Z_2$ type that are protected by time reversal symmetry. The axial part of the torsion will couple to the three dimensional Dirac fermions, break the TRS and affect the topological stability. The density of states associated to the screw dislocations in [50] can be reproduced with the techniques outlined in this work and will be reported elsewhere [61].

## IV. DISLOCATIONS IN CONVENTIONAL ELECTRONIC SYSTEMS

Another interesting question is how dislocations affect the electronic properties of crystal systems whose quasi-particles are described by the conventional Schrodinger equation. Within the present formalism the question can be posed as: Do scalar fields see torsion?

The question is interesting and relevant since the majority of electronic systems (Fermi liquids) obey a conventional Schrodinger equation and can be modelled with scalar fields. The point to notice here is that, although scalar fields transform in quite a trivial manner under rotations and translations, building a proper Hamiltonian means building derivatives and there is where curvature and perhaps torsion plays a role. The Lagrangian for a scalar field in a curved space is

$$\mathcal{L}_0 = \int d^n x \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \qquad (11)$$

As we see it involves only the metric and hence it does not see the torsion. In the spirit of building phenomenological couplings of the electronic density to the dislocations the

most natural scalar couplings that we can form from the torsion vectors (7) are

$$\mathcal{L}_{\text{int}} = \int d^n x g_1 (\partial_\mu V^\mu) \phi^2 + g_2 V^\mu V_\mu \phi^2, \qquad (12)$$

where $V_\mu$ is the vector associated to the trace of the torsion, i. e., to the density of edge dislocations. The coupling $g_2$ being quadratic in the torsion will be suppressed with respect to $g_1$ by an extra power of the density of dislocations. In the two dimensional case we can form the same interactions with the scalar field of (9).

Landau levels in the presence of a screw dislocation have been explored with a similar formalism in [29].

## V. CONCLUSIONS AND DISCUSSION

The proposal of modelling dislocations in a crystal with a geometrical approach including torsion within the elasticity theory is very old. What is new in our approach is the suggestion to study the influence of dislocations on the electronic properties of graphene and other electronic systems having Dirac fermions in the spectrum, by coupling the Dirac equation to a generalized curved space with torsion. In what two dimensional graphene sheets are concerned the main results of this work are somehow disappointing: spinors in two spatial dimensions do not couple to the axial part of the torsion (since there are no screw dislocations). Such a coupling would have had very important consequences. The coupling to the trace part is in the form of a vector gauge field similar to the one produced by the curvature or by strain in a more conventional approach. The spacial distribution of this new gauge field will nevertheless depend on the density of dislocations.

The most interesting applications of the formalism outlined here will be in the area of topological insulators that often support massless chiral fermions in three dimensions. In [51] it was shown that the coupling of spinors to the axial vector induced by torsion generates at higher order a four Fermi interaction on the fermions what had an influence on neutrino oscillations. In the context of condensed matter such a coupling would renormalize the Hubbard interaction or generate it if it was absent. The presence of the torsion axial vector in the Dirac equation will have also implications on various fundamental aspects of the physics of condensed matter system described by Dirac fermions. As it breaks time reversal symmetry, it can prevent weak localization. As for the conductivity, the disorder induced by a random distribution of dislocations may change the universality class of previous models based on vacancies or other T-preserved disorder. The breakdown of time reversal symmetry will also lift the quantum protectorate associated to the Fermi points [38] and allow the opening of a gap although the torsion itself will not explain the observations reported on gaps in graphene [52, 53]. The physical implications

of the effective gauge fields induced by dislocations on graphene have been discussed in [54, 55, 56, 57].

## APPENDIX A: THE DIRAC EQUATION IN A SPACE WITH TORSION AND CURVATURE

### 1. Dirac equation in a curved space without torsion

The behavior of spinors in curved spaces is more complicated than that of scalar or vector fields because their Lorentz transformation rules do not generalize easily to arbitrary coordinate systems [58]. Instead of the usual metric $g_{\mu\nu}$ we must introduce at each point $X$ described in arbitrary coordinates, a set of locally inertial coordinates $\xi_X^a$ and the vielbein fields $e_\mu^a(x)$, a set of orthonormal vectors labelled by $a$ that fixes the transformation between the local and the general coordinates:

$$e_\mu^a(X) \equiv \frac{\partial \xi_X^a(x)}{\partial x^\mu}\Big|_{x=X}. \qquad (A1)$$

The curved space gamma matrices $\gamma^\mu(x)$ satisfying the commutation relations

$$\{\gamma_\mu \gamma_\nu\} = 2g_{\mu\nu}, \qquad (A2)$$

are related with the constant, flat space matrices $\gamma^a$ by

$$\gamma^\mu(x) = e_a^\mu \gamma^a. \qquad (A3)$$

The spin connection $\Omega_\mu(x)$ is defined from the fielbein by

$$\Omega_\mu(x) = \frac{1}{4} \gamma_a \gamma_b e_\lambda^a(x) g^{\lambda\sigma}(x) \nabla_\mu e_\sigma^b(x), \qquad (A4)$$

where $\nabla_\mu$ is the covariant derivative acting on the $e_\mu$ vectors as

$$\nabla_\mu e_\sigma^a = \partial_\mu e_\sigma^a - \Gamma_{\mu\sigma}^\lambda e_\lambda^a. \qquad (A5)$$

The spin connection for 1/2 spinors can be written as

$$\Omega_\mu = \frac{1}{8} \omega_\mu{}^{ab} [\gamma_a, \gamma_b], \qquad (A6)$$

where $\omega_\mu{}^{ab}$ are the spin connection coefficients

$$\omega_\mu{}^{ab} = e_\nu^a \left(\partial_\mu + \Gamma_{\mu\lambda}^\nu\right) e^{b\lambda}. \qquad (A7)$$

In the flat case $\Gamma_{\mu\sigma}^\lambda$ is the usual affine connection (Christoffel symbols) related to the metric tensor by

$$\Gamma_{\mu\sigma}^\lambda = \frac{1}{2} g^{\nu\lambda} \{\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\nu}\}. \qquad (A8)$$

Finally, the determinant of the metric needed to define a scalar density lagrangian is given by

$$\sqrt{-g} = [\det(g_{\mu\nu})]^{1/2} = \det[e_\mu^a(x)]. \qquad (A9)$$

These formulas apply to a curved space without torsion. The Dirac equation is then written as

$$i\gamma^\mu D_\mu \psi = 0, \qquad (A10)$$

where $D_\mu = \partial_\mu + \Omega_\mu$ is the covariant derivative acting on the spinors in the curved space and $\gamma^\mu = \gamma^a e_a^\mu$ the curved gamma matrices.

## 2. Non-singular torsion in differential geometry

In differential geometry we consider a curved manifold, described by a set of coordinates $x^\mu$. To this manifold we associate a metric $g_{\mu\nu}$ which defines the distance locally. A notion of parallel transport is also needed to compare vectors in different points, in order to construct meaningful derivatives. The connection and the metric are independent geometric objects that are usually related by the so-called metricity condition ensuring that the relative angles and magnitudes of vectors are preserved under parallel transport. The metricity condition reads:

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\rho\mu} = 0, \qquad (A11)$$

this is, the covariant derivative of the metric vanishes. The metricity condition fixes the symmetric part of the connection and leaves the antisymmetric part undetermined. The most general connection, compatible with the Riemannian metric, is

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu \ \nu \end{matrix} \right\} + K^\lambda{}_{\mu\nu}, \qquad (A12)$$

where: $\left\{ \begin{matrix} \lambda \\ \mu \ \nu \end{matrix} \right\}$ are the Christoffel symbols and $K^\lambda{}_{\mu\nu}$ is called the contortion tensor related with the torsion tensor by

$$K^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} (T_{\beta\mu\nu} + T_{\mu\beta\nu} + T_{\nu\beta\mu}). \qquad (A13)$$

The torsion $T^\lambda{}_{\mu\nu}$ is the antisymmetric part of the connection:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \qquad (A14)$$

This means that when choosing a connection for a metric space, we are always free to choose the antisymmetric part of the connection (the torsion) at will. Only the symmetric part is fixed by the metricity condition. The usual choice is to take the torsion equal to zero, so that the metric alone determines all geometric properties. This is, for example, the case of General Relativity. The only restriction on K is that

$$K_{\lambda\mu\nu} = -K_{\nu\mu\lambda} \qquad (A15)$$

hence in a d-dimensional space it has $\frac{1}{2}[d^2(d-1)]$ components. The differential geometry of a general curved space is determined by the two tensor fields $g_{\mu\nu}$ and $T^\alpha_{\mu\nu}$ (or, equivalently $K^\alpha_{\mu\nu}$).

In a space with torsion the shortest path (geodesics) and the covariantly paralell path do not coincide and, as was mentioned in Section II, parallelograms do not close.

## 3. Torsion in a 2D flat space

Aiming to a description of dislocations alone, we study here the special case of a space with torsion but no curvature, in 2+0 dimensions. This will be the spatial part of the space-time used to describe graphene with dislocations as the Dirac equation in a space with spatial torsion.

As we will see, life is very simple in this space. First of all, since there is no curvature, there is a global coordinate system in which the metric tensor is the identity. Or seen from the tetrad perspective, there is a coordinate system in which the tetrads are trivial, and thus the Lorentz connection and the Einstein connection are equal. (Since the tetrads are trivial, there is no difference between latin and greek indices now, and moreover, since we are in 2+0, upper and lower indices are equal.) In this particular case, with the tetrads chosen in this way, we can say that the torsion in encoded in the connection.

Moreover, the antisymmetry of the torsion tensor in 2D immediately implies that it can be written as:

$$T^\lambda_{\mu\nu} = \epsilon_{\mu\nu} b^\lambda, \qquad (A16)$$

where $b^\lambda$ is an arbitrary vector field, basically the density of Burgers vector. It could be thought that this vector has some further constraints due to the Bianchi identities [28], but it can be checked that in 2D if $R_{\mu\nu\rho\sigma} = 0$ these identities are trivial.

## 4. Dirac equation in a curved space with torsion

As have seen in appendix A 2, inclusion of torsion in the geometrical description simply changes the connection and hence the covariant derivative.

It would be tempting then to propose that the Dirac equation in the space with torsion and curvature is given by eq. (5) with the appropriate connection but there are some important subtleties that we will specify [25, 59].

Consider the Dirac field in flat space. The massless, manifestly hermitian Dirac Langrangian may be written as:

$$\mathcal{L} = \int d^n x \frac{1}{2} \left[ \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi \right]. \qquad (A17)$$

Noting that:

$$\left( \partial_\mu \bar{\psi} \right) \gamma^\mu \psi = \partial_\mu \left[ \bar{\psi} \gamma^\mu \psi \right] - \bar{\psi} \gamma^\mu \partial_\mu \psi, \qquad (A18)$$

plus the fact that a total derivative doesn't affect the equations of motion (with suitable boundary conditions), the Lagrangian (A17) can be traded for:

$$\mathcal{L} = \int d^n x \bar{\psi} \gamma^\mu \partial_\mu \psi. \qquad (A19)$$

Indeed, the Dirac equation can be derived by variation with respect to $\bar{\psi}$ of any of those Lagrangians. The simpler form (A19) seems to be not hermitian, but this is not a problem since we can always introduce the boundary term back.

Now consider the same problem in a manifold with curvature and torsion. Consider the hermitian Lagrangian:

$$\mathcal{L} = \int d^n x \sqrt{g} \frac{1}{2} \left[ \bar{\psi}\gamma^\mu D_\mu \psi - \left( D_\mu \bar{\psi} \right) \gamma^\mu \psi \right], \quad \text{(A20)}$$

with $D_\mu = \partial_\mu - \omega_{\mu bc}\gamma^b\gamma^c$. Now the equivalent of the chain rule is:

$$\left( D_\mu \bar{\psi} \right) \gamma^\mu \psi = D_\mu \left[ \bar{\psi}\gamma^\mu \psi \right] - \bar{\psi} D_\mu \left( \gamma^\mu \psi \right). \quad \text{(A21)}$$

Note the following important point: since the covariant derivative contains gamma matrices, we can not just commute it to make it act directly on the spinor field. The "chain rule" that we have to use is really:

$$\left( D_\mu \bar{\psi} \right) \gamma^\mu \psi = D_\mu \left[ \bar{\psi}\gamma^\mu \psi \right] - \bar{\psi}\gamma^\mu D_\mu \psi + 4\omega_{\mu bc}e^{\mu b}\bar{\psi}\gamma^c\psi, \quad \text{(A22)}$$

where we have used:

$$\gamma^b\gamma^c\gamma^a = \gamma^a\gamma^b\gamma^c + 2\left( \gamma^b\eta^{ac} - \gamma^c\eta^{ab} \right). \quad \text{(A23)}$$

The commutation with the gamma matrices has introduced a new term, the trace of the connection. This means that the equivalence of the simpler form given in eq. (A19) is not just obtained by promoting the derivative to a covariant one, but it also requires the introduction of this trace [59].

In four dimensions, the following identity:

$$\gamma^a\gamma^b\gamma^c = \gamma^a g^{bc} + \gamma^c g^{ab} - \gamma^b g^{ac} + i\epsilon^{abcd}\gamma^5\gamma^d, \quad \text{(A24)}$$

and the definition of the connection (A6) allows to rewrite this as:

$$\mathcal{L} = \int d^4 x \left( \bar{\psi}\gamma^a\partial_a\psi + \frac{T_{abc}}{4}\bar{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi \right), \quad \text{(A25)}$$

or:

$$\mathcal{L} = \int d^4 x \left( \bar{\psi}\gamma^a\partial_a\psi + i\epsilon^{abcd}T_{bcd}\bar{\psi}\gamma^5\gamma^a\psi \right), \quad \text{(A26)}$$

Which reveals the well known result that, in General Relativity, fermions only couple to the antisymmetric part of the torsion [59]. In three dimensions using the identity:

$$[\gamma^a, \gamma^b] = -2\epsilon^{abc}\gamma_c, \quad \text{(A27)}$$

we can write the action in the form

$$\mathcal{L} = \int d^3 x \left( \bar{\psi}\gamma^a\partial_a\psi + i\epsilon^{abc}T_{abc}\bar{\psi}\psi \right), \quad \text{(A28)}$$

from where the Dirac equation can be extracted directly to read

$$[\gamma^a\partial_a + i\epsilon^{abc}T_{abc}]\psi = 0, \quad \text{(A29)}$$

*Note added*: After completing this work we became aware of ref. [60] where similar issues are discussed. We think that, although there is some obvious overlapping, the two works entertain different points of view and can be seen as complementary.

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